ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, No. 3, 1978

MINIMAL DEGREE FOR A PERMUTATION REPRESENTATION OF A CLASSICAL GROUP

BY

BRUCE N. COOPERSTEIN[†]

ABSTRACT

The minimal degree for a permutation representation of the finite linear groups, and finite classical groups is determined.

§ 1. Introduction

The purpose of this paper is to determine the minimal degree of a permutation representation of a classical group. Interest in this type of problem dates back to Galois. Galois proved if $G = \mathrm{PSL}_2(p)$, p a prime, then the minimal degree of a permutation representation of G is $p+1$ provided $p > 11$, while the degree is p if $p \leqq 11$. In his thesis [15], W. Patton determined the minimal degree for $G = \mathrm{SL}_n(q)$ or $\mathrm{SP}_{2n}(q)$, q odd. We determine the minimal degree for the remaining classical groups. We also include in our proof Patton's results as they have not been published.

§2. Minimal degrees of the classical groups

Table 1 gives the degree $\mathrm{d}(G)$ of a permutation representation of a linear or classical group G which we assert is the minimal degree for G. In the third column we have indicated the stabilizer of a point in this representation.

§ 3. Root elements in linear and classical groups

Let V be a vector space of dimension $n \geqq 2$ over $\mathbf{F}_q = \mathbf{F}_{p^e}$. For $T \in \mathrm{GL}(V)$, $W \leqq V$, denoted by $[T, W] = [W, T]$ the subspace of V generated by $Tw - w$ as w runs over W. When Q is a point (i.e., one-space) of V and H is a hyperplane

[†]Supported in part by NSF MCS 76–07035.
Received February 3, 1977 and in revised form July 8, 1977

TABLE 1

G	d(G)	Point Stabilizer
$\mathrm{SL}_n(q)$, $(n,q) \neq (2,2), (2,3)$, $(2,5), (2,7), (2,9), (2,11), (4,2)$	$\frac{q^n-1}{q-1}$	Stabilizer of a point in usual 2-transitive representation on points or hyperplanes of $\mathrm{PG}(n-1,q)$
$\mathrm{SL}_2(2)$	2	A three Sylow $\cong \mathbf{Z}_3$
$\mathrm{SL}_2(3)$	3	A 2-Sylow $\cong Q_8$
$\mathrm{SL}_2(5)$	5	A 2-Sylow normalizer $\cong \hat{A}_4$
$\mathrm{SL}_2(7)$	7	Either of the 2-classes of $\hat{\Sigma}_4$
$\mathrm{SL}_2(9)$	6	$\hat{A}_5$ (2 classes)
$\mathrm{SL}_2(11)$	11	$\hat{A}_5$
$\mathrm{SL}_4(2)$	8	A_7
$\mathrm{Sp}_4(2)$	2	A_6
$\mathrm{Sp}_{2n}(2) \cong \Omega_{2n+1}(2)$, $n \geqq 3$	$2^{n-1}(2^n-1)$	$\mathrm{GO}^-_{2n}(2)$
$\mathrm{Sp}_{2n}(q)$, $n \geqq 2$, $q > 2$, $(n,q) \neq (2,3)$	$\frac{q^{2n-1}}{q-1}$	Stabilizer of a point of $\mathrm{PG}(2n-1,q)$
$\mathrm{Sp}_4(3)$	27	A maximal 2-local which is a split extension of $\mathbf{F}_2^4$ by $\Omega_4^-(2) \cong A_5$
$\Omega_{2n+1}(q)$, $n \geqq 3$, q odd	$q^{2n}-1/q-1$	Stabilizer of a singular point of $\mathrm{PG}(2n,q)$
$\Omega^+_{2n}(q)$, $n \geqq 4$, $q > 2$	$\frac{(q^n-1)(q^{n-1}+1)}{q-1}$	Stabilizer of a singular point of $\mathrm{PG}(2n-1,q)$
$\Omega^+_{2n}(2)$, $n \geqq 4$	$2^{n-1}(2^n-1)$	Stabilizer of a non-singular point of $\mathrm{PG}(2n-1,2)$, $\Omega_{2n-1}(2) = \mathrm{SP}_{2n-2}(2)$
$\mathrm{SU}_3(q)$, $q \neq 2,5$	q^3+1	Stabilizer of an absolute point of $\mathrm{PG}(2,q^2)$, normalizer of a p-Sylow, $p = \mathrm{char}\,\mathbf{F}_q$
$\mathrm{SU}_3(2)$	2	Centralizer of an element of order 3-extention of a three group of order 27 by Z_4
$\mathrm{SU}_3(5)$	50	$\hat{A}_7$-3 fold cover of A_7

$SU_4(q)$	$(q+1)(q^3+1)$	Stabilizer of a totally isotropic line in $PG(3,q^2)$; a maximal p-local which is a split extension of F_q^4 by $\Omega_4^-(q) \cong SL_2(q^2)$
$SU_n(q)$, $n \geqq 5$	$\frac{[q^n-(-1)^n][q^{n-1}-(-1)^{n-1}]}{q^2-1}$	Stabilizer of an absolute point of $PG(n-1,q^2)$
$\Omega_{2n}^-(q)$, $n \geqq 4$	$\frac{(q^n+1)(q^{n-1}-1)}{q-1}$	Stabilizer of a singular point of $PG(2n-1,q)$

with $Q \subseteq H$, then $T \in GL(V)$ is said to be a transvection with center Q and axis H if $[T, V] = Q$, $[T, H] = 0$. $SL(V)$ is the subgroup of $GL(V)$ consisting of those transformations of determinant 1. In all cases $SL(V) = \langle T : T$ a transvection$\rangle$. Set $G = SL(V)$. A parabolic subgroup of G is the stabilizer of some chain of subspaces. The maximal parabolic subgroups are the stabilizers of some subspace W (a chain of length 1), G_W, and these are the maximal p-local subgroups of G. G is transitive on chains of a given type, and in particular on subspaces of a given dimension. If $L_k(V)$ denotes the collection of k-subspaces of G, then for $W \in L_k(V)$,

$$|G : G_W| = |L_k(V)| = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=1}^{n}(q^j-1)}{\prod_{j=1}^{k}(q^j-1)\cdot\prod_{j=1}^{n-k}(q^j-1)}.$$

Then clearly $|G : G_{\mathscr{C}}| \geqq (q^n-1)/(q-1)$ for any chain of subspaces. In particular, if $X \lneqq G$ so $|G : X| < d(G)$, then X acts irreducibly on V.

Now suppose V is equipped with a non-degenerate alternate or skew hermitian form f. For uniformity we denote by $GO(f)$ the group of isometries of $f : GO(f) = \{T \in GL(V) : f(Tv, Tw) = f(v, w)$ for all $v, w \in V\}$. $SO(f) = GO(f) \cap SL(V)$. When f is alternate set $q_0 = q$ and when f is skew-hermitian, let $q_0 = \sqrt{q}$. It is well-known if $\dim V = 2$, then $SO(f) \cong SL_2(q_0)$. Thus we assume $\dim V \geqq 3$. A vector v in V is said to be isotropic if $f(v, v) = 0$, and then we say the point $\langle v\rangle$ is absolute. For $W \leqq V$, set $W^\perp = \{v \in V : f(v, w) = 0$ for all $w \in W\}$. If v is an isotropic vector, then $T_v : V \to V$ given by

$$T_v(w) = w - f(w, v)v \quad \text{is in } SO(f).$$

In fact unless f is skew-hermitian, $\dim V = 3$ and $q = 4$, $SO(f) = \langle T_v : f(v, v) = 0\rangle$. When f is skew hermitian, $\dim V = 3$, $q = 4$, then $SO(f)$ is a group of order $2^3 \cdot 3^3$ and $\langle T_v : f(v, v)\rangle$ is a normal subgroup of index 4. Note that T_v is a transvection with center $\langle v\rangle$ and axis $\langle v\rangle^\perp$. A subspace W of V is totally isotropic if $W \subseteq W^\perp$. The parabolic subgroups are the stabilizer of chains of totally isotropic subspaces, and the maximal parabolics are the stabilizers of totally isotropic subspaces. When f is alternate we denote the group GO(f) by Sp(f), Sp(V), $\mathrm{Sp}_n(q)$. We remark in the alternate case GO(f) = Sp(f). When f is skew-hermitian we denote GO(f) by GU(f), GU(V), etc., and SO(f) by SU(f), etc. SO(f) is transitive on chains of totally isotropic subspaces of any given type (Witt's theorem), and in particular on totally isotropic subspaces of a given dimension. Suppose $G = \mathrm{Sp}(V)$ where $\dim V = n = 2m$, and W is a totally isotropic k-space. Then

$$(3.1)\qquad |G:G_W| = \frac{\prod_{j=1}^{m}(q^{2j}-1)}{\prod_{j=1}^{k}(q^{j}-1)\cdot\prod_{j=1}^{m-k}(q^{2j}-1)}.$$

Note that this is at least d(G). When $q \neq 2$ and $(m,q)\neq(2,3)$, W an isotropic one-space, see that $|G:G_W| = \mathrm{d}(G)$. Consequently the minimal degree for G is at most d(G) in these cases. Suppose $X \leqq G$, $|G:X| < \mathrm{d}(G)$ and X is not irreducible on V. Let W be a minimal X-invariant subspace of V. Then $\mathrm{Rad}\, W = W \cap W^\perp$ is also X-invariant, $\mathrm{Rad}\, W \leqq W$. If $\mathrm{Rad}\, W = W$, then W is totally isotropic and $|G:X| \geqq |G:G_W| \geqq \mathrm{d}(G)$, contradicting our assumption that $|G:X| < \mathrm{d}(G)$. Therefore $W \cap W^\perp = 0$, W is non-degenerate and $V = W \oplus W^\perp$. The map $g \to (g\mid W, g \mid W^\perp)$ is an isomorphism from X to $\mathrm{Sp}(W)\times \mathrm{Sp}(W^\perp)$. If $\dim W = 2k$, then

$$|G:X| \geqq q^{k(2m-k)}\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} > \mathrm{d}(G).$$

Therefore, if $X \leqq G$ so $|G:X| < \mathrm{d}(G)$, then X acts irreducibly on V.

Now suppose $G = \mathrm{SU}(V) \cong \mathrm{SU}_n(q_0)$, and W is a totally isotropic k-space. Then

$$(3.2)\qquad |G:G_W| = \frac{\prod_{j=0}^{2k-1}[q_0^{n-j}-(-1)^{n-j}]}{\prod_{j=1}^{k}(q_0^{2j}-1)}.$$

In all cases this is at least d(G). Suppose $n = 3$ or $n \geqq 5$ and $(n, q_0) \neq (3,2)$ or $(3,5)$. If W is an absolute point, then $|G:G_W| = \mathrm{d}(G)$, so that the minimal degree in these cases is at most d(G). When $n = 4$, let W be a totally isotropic 2-subspace of V. Then $|G:G_W| = \mathrm{d}(G)$, so in this case as well the minimal degree is no greater than $d(G)$. Now assume $X \leqq G$ is reducible on V. Take W to be a minimal X-invariant subspace of V. Then $W = \mathrm{Rad}\, W$, or $0 = \mathrm{Rad}\, W$. In the first case $|G : X| \geqq |G : G_W| \geqq d(G)$ by the remark following (3.2). Suppose $\mathrm{Rad}\, W = 0$, so $V = W \oplus W^{\perp}$. Then $|G_w| = |\mathrm{GU}(W)| \cdot |\mathrm{SU}(W^{\perp})| = (q_0 + 1)|\mathrm{SU}(W)| \cdot |\mathrm{SU}(W^{\perp})|$. If $\dim W = k$, then

$$|G : X| \geqq |G : G_W| = \frac{q_0^{(n-k)k} \cdot \prod_{j=2}^{n} [q_0^j - (-1)^j]}{\prod_{j=1}^{k} [q_0^j - (-1)^j] \cdot \prod_{j=1}^{n-k} [q^j - (-1)^j]}$$

and then $|G : X| > d(G)$.

Now for $G = \mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SU}(V)$, the transvections in G will be called the long root elements of G. We denote by Γ the collection of all cyclic subgroups of G generated by a long root element. For $x \in \Gamma$, $R_x = Z(O^{p'}(C_G(x)))$, and this is the long root subgroup of G containing x. $\mathscr{X} = \{R_x : x \in \Gamma\}$. For a subgroup X of G, $\Gamma(X) = \{x \in \Gamma : x \leqq X\}$. Also if $x \in \Gamma$, let V_x be the center of x (i.e. $[V, x]$). Note when $G = \mathrm{Sp}(V)$ or $\mathrm{SU}(V)$, then for $x, y \in \Gamma$, $[x, y] = 1$ if and only if $V_x \leqq V_y^{\perp}$. Before turning our attention to orthogonal groups we point out that $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, and consequently the minimal degree for $\mathrm{Sp}_4(3)$ is no greater than $d(G) = 27$.

Now let V be equipped with a non-degenerate quadratic form Q, and let f be the associated symmetric form. A vector v is singular if $Q(v) = 0$. A subspace is totally singular if all its vectors are singular. For W a subspace of V we denote by $W^{\perp} = \langle v \in V : f(v, w) = 0 \rangle$, and $\mathrm{Rad}\, W = W \cap W^{\perp}$. Note that $\mathrm{Rad}\, V = 0$ unless $\dim V$ is odd and $p = 2$, in which case $\dim \mathrm{Rad}\, V = 1$ and $Q(\mathrm{Rad}\, V) \neq 0$. The isometry group of Q, $\{T \in \mathrm{GL}(V) : Q(v) = Q(Tv)$ for all $v \in V\}$ is denoted by $\mathrm{GO}(V)$. Let $W = \langle v_1, v_2 \rangle$ be a totally singular two-subspace of V. Then the element of $\mathrm{GL}(V)$ defined by

$$T(w) = w - f(v_1, w)v_2 + f(v_2, w)v_1$$

is in $\mathrm{GO}(V)$. Such an element is called a long root element. $\Omega(V)$ is the group generated by long root element. When $G = \Omega(V)$ we denote by Γ the collection of cyclic subgroups of G generated by a long root element. When dimension of V, n, is odd, $n = 2m + 1$, we also denote $\Omega(V)$ by $\Omega_{2m+1}(V)$, when dimension of V is even, $n = 2m$, we denote $\Omega(V)$ by $\Omega_{2m}^{+}(q)$ on $\Omega_{2m}^{-}(q)$ as the index of f is maximal or non-maximal. Note that $\Omega_2^{\varepsilon}(q)$ is a cyclic group of order $q - \varepsilon$, $\Omega_3(q) \cong \mathrm{SL}_2(q)$, $\Omega_4^{+}(q) \cong \mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$, $\Omega_4^{-}(q) \cong \mathrm{SL}_2(q^2)$, $\Omega_5(q) \cong \mathrm{Sp}_4(q)$, $\Omega_6^{+}(q) \cong \mathrm{SL}_4(q)$ and $\Omega_6^{-}(q) \cong \mathrm{SU}_4(q)$. Also $\Omega(V)$ is irreducible on V unless dimension of V is odd and q is even. In this case $\Omega(V)$ normalizes Rad V and acts as a symplectic group on $V/\text{Rad } V$.

The parabolic subgroups of $\Omega(V)$ are the stabilizers of chains of totally singular subspaces, and the maximal parabolics are the stabilizers of totally singular subspaces. Suppose $\dim V = 2m + 1$, and W is a totally singular subspace of dimension k. Then

$$|G : G_W| = \frac{\prod_{j=1}^{m} (q^{2j} - 1)}{\prod_{j=1}^{k} (q^j - 1) \cdot \prod_{j=1}^{m-k} (q^{2j} - 1)}. \tag{3.3}$$

Thus $|G : G_W| \geqq d(G)$ for every totally singular subspace. Note if q is odd and W is a singular one-space, then $|G : G_W| = d(G)$, so that in this case the minimal degree of G is no greater than $d(G)$. If q is even, then it is implied above that $G \cong Sp_{2m}(q)$. We already have remarked that if $q > 2$, then the minimal degree of G is at least $d(G)$. Suppose that $q = 2$. G contains $GO^{\varepsilon}_{2m}(2)$ and the index of this group in G is $2^{m-1}(2^m + \varepsilon)$. Consequently the minimal degree of G is no greater than $d(G)$ in this case. Suppose $X \leqq G$ and X acts faithfully on $V/\text{Rad } V$. Let W be a minimal X invariant subspace containing Rad V. If Rad $W \neq$ Rad V, then Rad $W = W$ and X is contained in a parabolic subgroup, so $|G : X| \geqq d(G)$. If Rad $W =$ Rad V, then the map from X to $GO(W) \times GO(V/W)$ by $g \to (g|w, g|v/w)$ is an injection. It then follows by computing the orders of $GO(W)$ and $GO(V/W)$ that $|G : X| \geqq d(G)$. Thus if G has a subgroup X so $|G : X| < d(G)$, then X acts irreducibly on $V/\text{Rad } V$.

Now consider $G = \Omega^{\varepsilon}_{2m}(q)$. As before the parabolic subgroups are the stabilizers of chains of totally singular subspaces, and are maximal if and only if the chain has length one. Suppose W is a totally singular k-subspace of V. Then

$$|G : G_W| = \frac{\prod_{j=0}^{k-1} (q^{m-2j} - \varepsilon)(q^{m-2j-1} + \varepsilon)}{\prod_{j=1}^{k} (q^j - 1)} \tag{3.4}$$

and so $|G : G_W| \geqq d(G)$ for any totally singular subspace W. Suppose either $\varepsilon = -1$ or $q > 2$, W a singular one-space. Then $|G : G_W| = (q^m - \varepsilon)(q^m + \varepsilon)/q - 1 = d(G)$, so the minimal degree of G in these cases is no greater than $d(G)$. Thus consider $\Omega^{+}_{2m}(2) = G$. G acts transitively on the nonsingular vectors in V (Witt–Dieudonne Theorem), and there are $2^{2m} - 1 - (2^m - 1)(2^{m-1} + 1) = 2^{m-1}(2^m - 1) = d(G)$ such vectors. The centralizer of such a vector w is isomorphic to $\Omega(\langle w \rangle^{\perp}) \cong \Omega_{2m-1}(2) \cong Sp_{2m-2}(2)$. Thus in this case as well the minimal degree of G is no greater than $d(G)$. As in the previous cases we can show if $X \leqq G$ acts reducibly on V, then $|G:X| \leqq d(G)$.

Before we get to the remaining results of this section we introduce some notation for the orthogonal groups. Recall Γ is the collection of cyclic subgroups of $G = \Omega(V)$ generated by a long root element. For $x \in \Gamma$, let $R_x = Z(O^{p'}(C_G(X)))$, so that R_x is the long root subgroup containing x, $\mathscr{X} = \{R_x : x \in \Gamma\}$, let $V_x = [V, x]$. If $T \in x$ then there is a base v_1, v_2 for V_x so

$$T(w) = w - f(v_1, w)v_2 + f(v_2, w)v_1.$$

Also, for a subgroup H of G we write $\Gamma(H)$ to denote the subset of Γ of subgroups contained in H.

(3.5) LEMMA. *Let G be a group of Lie type of rank at least two over $\mathbf{F}_q$, $q = p^e$, G not ${}^2F_4(q)$. Let R be a long root subgroup, $T = Z(R)$. Set $\mathscr{X} = T^G$. For $H \leqq G$, $\mathscr{X}(H) = \{X \in \mathscr{X} : X \leqq H\}$. For $X, Y \in \mathscr{X}$ one of the following hold:*

(i) $[X, Y] = 1$, $|\mathscr{X}(\langle X, Y\rangle)| = q + 1$, $\langle X, Y\rangle^{\#} = \bigcup_{Z \in \mathscr{X}(\langle X, Y\rangle)} Z^{\#}$. *In this case write* $(X, Y) \in \Delta$.

(ii) $[X, Y] = 1$, $\mathscr{X}(\langle X, Y\rangle) = \{X, Y\}$ *and we write* $(X, Y) \in \Delta_2$.

(iii) $|\langle X, Y\rangle| = q^3$, $\langle X, Y\rangle' = [X, Y] = Z(\langle X, Y\rangle) \in \mathscr{X}$, *and* $(X, [X, Y])(Y, [X, Y]) \in \Delta$. *We write* $(X, Y) \in \Delta_3$.

(iv) $\langle X, Y\rangle \cong SL_2(q)$ *or* $PSL_2(q)$ *and we write* $(X, Y) \in \Delta_4$.

PROOF. See (12.1) of [1].

(3.6) LEMMA. *Let* $(X, Y) \in \Delta$, $Z \in \Delta_4(X) = \{Z \in \mathscr{X} : (X, Z) \in \Delta_4\}$. *Then* $|\mathscr{X}(\langle X, Y\rangle) \cap \Delta_3(Z)| = 1$, $\mathscr{X}(\langle X, Y\rangle) - \Delta_3(Z) \subseteq \Delta_4(Z)$.

PROOF. When G is of type A_l ($G = SL_{l+1}(q)$) this is straightforward to check. When G is of type C_l or ${}^2A_l$ there are no pairs of type Δ, so there is nothing to prove. In the remaining cases, the representation $(G, \mathscr{X})$ is a maximal parabolic representation. By [2], there is a distinguished self-paired orbital so that the incidence structure associated with this orbital is thick. The orbital is always Δ, and for $(X, Y) \in \Delta$, the line on X and Y is $\mathscr{X}(\langle X, Y\rangle)$. Moreover, in each of these geometries, $O_p(G_X) = O_p(N_G(X))$ is regular on $\Delta_4(X)$ and $G_{X,Z} \geqq C_G(\langle X, Z\rangle)$ is transitive on the lines on X, where $Z \in \Delta_4(X)$. Therefore, it suffices to prove the assertion for a single line on X and a single $Z \in \Delta_4(X)$. In each case this is a straightforward calculation.

(3.7) NOTATION. If X is an elementary abelian group, let $\mathscr{E}_1(X)$ be the collection of its non-trivial cyclic subgroups.

Let $G, \mathscr{X}$ be as in (3.5) and let $\Gamma = \bigcup_{X \in \mathscr{X}} \mathscr{E}_1(X)$. For a subgroup H of $G, \Gamma(H) = \{x \in \Gamma : x \leqq H\}$. Define $\sim$ on Γ by $u \sim v$ if and only if u and v are conjugate in $\langle u, v \rangle$. We remark that if we set $R_x = Z(O^{p'}(C_G(X)))$ for $x \in \Gamma$, then $R_x \in \mathscr{X}$, and for $u, v \in \Gamma$, $u \sim v$ if and only if $(R_u, R_v) \in \Delta_4$. For $H \leqq G$, $u \in \Gamma(H)$, H_u will denote the connected component of $(\Gamma(H), \sim | \Gamma(H))$ containing u, and $H(u) = \langle H_u \rangle$.

(3.8) LEMMA. *Assume* $H = \langle \Gamma(H) \rangle$, $O_p(H) = 1$.

(i) *If* $u, v \in \Gamma(H)$, $[u, v] = 1$ *and* $\mathscr{E}_1(\langle u, v \rangle) \subseteq \Gamma$, *then* $v \in H_u$.

(ii) *If* $v \in \Gamma(H) - H_u$, *then* $[v, H(u)] = 1$.

(iii) H_u *is a conjugacy class in* $H(u)$ *and* H.

PROOF. Since $O_p(H) = 1$, by a theorem of Baer (see Theorem 38.7 in [6]), there is a conjugate w of u so $\langle u, w \rangle$ is not a p-group, and a conjugate z of v so $\langle z, v \rangle$ is not a p-group. Then $u \sim w$ and $v \sim z$. If $R_u = R_v$, then $\langle w, v \rangle$ is not a p-group, and then $v \sim w$, so that $v \in H_u$. Therefore, we may assume $R_u \neq R_v$. Let $R_1, R_2, \cdots, R_{p+1}$ be the root subgroups that intersect $\langle u, v \rangle$ non-trivially, with $R_1 = R_u$, $R_2 = R_v$. Then for each $i \neq j$, $(R_i, R_j) \in \Delta$. Let $W = R_w$, $Z = R_z$. Then $(R_1, W) \in \Delta_4$, $(R_2, Z) \in \Delta_4$. If $(R_1, Z) \in \Delta_4$, or $(R_2, W) \in \Delta_4$ then one of u, z, v or u, w, v is a $\sim$ path from u to v, and then $v \in H_u$. So we may assume (R_1, Z), $(R_2, W) \not\in \Delta_4$. Then (R_1, Z), $(R_2, W) \in \Delta_3$. However, by (3.6) $|\{R_i : 1 \leqq i \leqq p+1\} \cap \Delta_3(Z)|$, $|\{R_i : 1 \leqq i \leqq p+1\} \cap \Delta_3(W)| \leqq 1$, and $\{R_i : 1 \leqq i \leqq p+1\} - \Delta_3(W)$ (resp. $\Delta_3(Z)) \subseteq \Delta_4(W)$ (resp. $\Delta_4(Z)$). Consequently $R_3 \in \Delta_4(W) \cap \Delta_4(Z)$. Set $r = \langle u, v \rangle \cap R_3$. Then u, w, r, z, v is a $\sim$ path from u to v, so $v \in H_u$ proving (i).

(ii) We claim if $v \in \Gamma(H) - H_u$, then $[u, v] = 1$. By (i) $\mathscr{E}_1(\langle u, v \rangle) \not\subseteq \Gamma$. If $\langle u, v \rangle$ is a p-group, $[u, v] \neq 1$, then if $w = [u, v]$ we have $\mathscr{E}_1(\langle u, w \rangle), \mathscr{E}_1(\langle v, w \rangle) \subseteq \Gamma$. Then $w \in H_u$, so $H_u = H_w$. Also $v \in H_w$, so $H_u = H_w = H_v$, contradicting $v \in \Gamma(H) - H_u$. Therefore $[u, v] = 1$ as asserted. Since for any $w \in H_u$, $H_u = H_w$, also $[w, v] = 1$. Then since $H(u) = \langle H_u \rangle$, $[H(u), v] = 1$.

(iii) Clearly all members of H_u are conjugated in $H(u)$, so it suffices to prove H_u is a conjugacy class in H. Suppose v is conjugate to u by an element g in H. Since $H = \langle \Gamma(H) \rangle$, there are $\langle x_1 \rangle, \cdots, \langle x_n \rangle \in \Gamma(H)$ so $g = x_1 x_2 \cdots x_n$. Set $u_0 = u$, $u_i = u^{x_1 \cdots x_i}$ for $1 \leqq i \leqq n$. Then $v = u_n$. We claim for each i, $H_{u_i} = H_{u_{i+1}}$. Clearly we may assume $u_i \neq u_{i+1}$. Then $[u_i, x_{i+1}] \neq 1$. If $u_i \sim x_{i+1}$, then also $u_i \sim u_i^{x_{i+1}} = u_{i+1}$ and $H_{u_i} = H_{u_{i+1}}$. If $u_i \not\sim x_{i+1}$, then $\langle u_i, x_{i+1} \rangle$ is a p-group (with order p^3). Set $w_{i+1} = [u_i, x_{i+1}]$. Then $\langle u_i, u_{i+1} \rangle = \langle u_i, w_{i+1} \rangle$. Moreover, by (3.5) (iii), $\mathscr{E}_1(\langle u_i, u_{i+1} \rangle) = \mathscr{E}_1(\langle u_i, w_{i+1} \rangle) \subseteq \Gamma$. By (i), $H_{u_i} = H_{u_{i+1}}$ and the lemma is completed.

(3.9) LEMMA. *Let the Weyl rank of G be m, and* $L = \langle R_x, R_y\rangle$ *where* $x \sim y$, *so that* $L \cong SL_2(q)$. *Let* $\mathcal{L} = L^G$. *Assume* $\mathcal{Y} \subseteq \mathcal{L}$ *so for* $L_1, L_2 \in \mathcal{Y}$, $[L_1, L_2] = 1$. *Then* $|\mathcal{Y}| \leqq m$.

PROOF. If $\mathcal{Y} = \{L_1, \cdots, L_t\}$, then $\langle L_i : i = 2, \cdots, t\rangle \leqq O^{p'}(C_G(L_1))$ which is a group of Lie type of rank $< m$. By induction $t - 1 \leqq m - 1$, so $t \leqq m$.

§4. Reductions

We assume in this section that G is a classical group, V the standard module for V, with the convention that if $G \cong SP_{2m}(2^e) \cong \Omega_{2m+1}(2^e)$, then V is the orthogonal module. Γ is the collection of cyclic subgroups generated by root elements of G. Assume that the result is false and choose a counterexample with $\dim V = n$ minimal. Let X be a proper subgroup of G with $|G : X| < d(G)$.

(4.1) LEMMA. *X is irreducible on $V/\mathrm{Rad}\ V$.*

PROOF. This was proved in §3.

(4.2) LEMMA. $U_G(X; p) = \{1\}$.

PROOF. If $U \in U_G(X; p)$, $U \neq 1$, then X normalizes $C_{V/\mathrm{Rad}\,V}(U) \neq 0$, $V/\mathrm{Rad}\ V$, contradicting (4.1).

(4.3) NOTATION. For $x \in \Gamma(X)$, set $W_x = \langle V_y : Y \in H_x\rangle$, $Y_x = \mathrm{Rad}\ W_x$.

(4.4) LEMMA. *Assume* $\Gamma(X) \neq \varnothing$. *Let* $x \in \Gamma(X)$, *and set* $H_1 = H(x)$. *Then* $|X : N_X(H_1)| \leqq [n/2]$.

PROOF. Note that the Weyl rank of G is at most $[n/2]$. $|X : N_X(H_1)|$ is the number of conjugates of H_1 in X, say t. Thus let the conjugates be $H_1, H_2, \cdots, H_t$ and $g_i \in X$ so $H_1^{g_i} = H_i$, $2 \leqq i \leqq t$. Choose $u \sim v \in H_x$ and set $L_1 = \langle R_u, R_v\rangle$, and $L_i = L_1^{g_i}$, $2 \leqq i \leqq t$. Then for $i \neq j$, $[L_i, L_j] = 1$. By (3.9), $t \leqq [n/2]$.

(4.5) LEMMA. *Suppose* $G \cong SU_n(q_0)$ *or* $Sp_{2m}(q)$ (*and in the latter case that q is odd*). *Assume* $\Gamma(X) \neq \varnothing$. *Then* $\Gamma(X)$ *is a conjugacy class in* $H = \langle \Gamma(X)\rangle$.

PROOF. Let $x \in \Gamma(X)$, and set $H_1 = H(X)$. We want to prove $H_x = \Gamma(X)$, so assume otherwise. We first prove $Y_x = \mathrm{Rad}\ W_x = 0$. Suppose $y \in \Gamma(X) - H_x$. Then $V_y \leqq V_x^\perp$, so $V_y \leqq W_x^\perp \leqq Y_x^\perp$. Therefore $Y_x \leqq \mathrm{Rad}\langle V_y : y \in \Gamma(X)\rangle$. However, $\mathrm{Rad}\langle V_y : y \in \Gamma(X)\rangle$ is normalized by X. By (4.1) X acts irreducibly on V, so $\mathrm{Rad}\langle V_y : y \in \Gamma(X)\rangle = V$ or 0. But $\mathrm{Rad}\langle V_y : y \in \Gamma(X)\rangle \neq V$, and so $Y_x = 0$ as asserted. Thus W_x is non-degenerate. We next claim $C_{W_x}(H_1) = 0$. For $y \in H_x$, $C_{W_x}(H_1) \leqq C_V(y) \leqq V_y^\perp$. Hence $C_{W_x}(H_1) \leqq \bigcap_{y \in H_x} = W_x^\perp$. Thus $C_{W_x}(H_1) \leqq$

$W_x \cap W_x^\perp = \operatorname{Rad} W_x = Y_x = 0$. Next we show H_1 acts irreducibly on W_x. Let U be an H_1-invariant subspace of W_x. As $[H_1, U] \neq 0$, there is a $y \in H_x$ so $[U, y] \neq 0$. Then $V_y = [U, y] \leqq U$. Since H_x is a conjugacy class in H_1, $U \geqq \langle V_y : y \in H_x \rangle = W_x$. Now if $y \in H_x$, $z \in \Gamma(X) - H_x$, then $V_y \leqq V_z^\perp$, so $[V_y, z] = 0$. Therefore $[W_x, z] = 0$. Now $N_X(H_1) \leqq N_X(W_x)$. Suppose $g \in N_X(W_x)$, $g \in H_x$. Then $[U, y^g] = [U, y]^g \neq 0$, so $y^g \in H_x$. Hence $N_X(W_x) \leqq N_X(H_1)$, so $N_X(H_1) = N_X(W_x)$. Let $\dim W_x = t < n$. Then $|X : N_X(W_x)| = n/t$. Now since W_x is non-degenerate, $|N_G(W_x)|$ is easily computed and $|G : N_G(W_x)| > d(G)n/t$, which contradicts $|G : X| < d(G)$.

(4.6) LEMMA. *Suppose G is an orthogonal group, $n \geqq 6$ and $\Gamma(X) \neq \emptyset$. Then $\Gamma(X)$ is a conjugacy class in $H = \langle \Gamma(X) \rangle$.*

PROOF. Let $x \in \Gamma(X)$, and set $H_1 = H(x)$. We must show $H_x = \Gamma(X)$, so we assume on the contrary that $\Gamma(X) \neq H_x$. We first show that $Y_x \leqq \operatorname{Rad}\langle V_z : z \in \Gamma(X) \rangle$. If we have shown this, then $Y_x \leqq \operatorname{Rad} V$, since X normalizes $\operatorname{Rad}\langle V_z : z \in \Gamma(X) \rangle$ and acts irreducibly on $V/\operatorname{Rad} V$. If $z \in \Gamma(X)$ and $V_x \cap V_z = 0$, $V_z \nleqq V_x^\perp$, then $(R_x, R_z) \in \Delta_3$ or Δ_4 and so by (3.8) $z \in H_x$. Then $Y_x = Y_z$ and so $Y_x \leqq V_z^\perp$. So if $V_x \cap V_z = 0$ we may assume $V_z \leqq V_x^\perp$, so then $V_z \leqq W_x^\perp \leqq Y_x^\perp$. Thus we may assume $V_x \cap V_z \neq 0$, $z \notin H_x$. Then since H_x is a conjugacy class in H_1, $[H_1, z] = 1$, for all $y \in H_x$, $V_y \cap V_z \neq 0$. But if $x \sim y$, then $V_x \cap V_y = 0$. Then since V_z is a two-space, $V_z = \langle V_x \cap V_z, V_y \cap V_z \rangle \leqq W_x$. Thus in this case $Y_x \leqq W_x^\perp \leqq V_z^\perp$ and so our claim is proved. Now set $\bar{V} = V/\operatorname{Rad} V$ and denote images in $V/\operatorname{Rad} V$ by $\bar{\ }$. We claim $\bar{W}_x = \bar{V}$. Note that if $y \in \Gamma(X)$ and $\bar{V}_y \cap \bar{W}_x \neq 0$, then $\bar{W}_y = \bar{W}_x$. Also as we pointed out above, if $V_x \cap V_y = 0$ and $V_y \nleqq V_x^\perp$, then $\bar{W}_x = \bar{W}_y$. Thus, if $\bar{W}_x \neq \bar{W}_y$, then $\bar{W}_y \leqq \bar{W}_x^\perp$. Therefore if $n' = \dim \bar{V}$ and $t = \dim \bar{W}_x$, since X is irreducible on $\bar{V}$, $|X : N_X(\bar{W}_x)| = n'/t$. Suppose $\bar{W}_x \neq \bar{V}$, so $t < n'$. Note that $t > 1$. Again it is easy to compute $N_G(\bar{W}_x)$ and $|G : N_G(\bar{W}_x)| > \mathrm{d}(G)n'/t$, from which it follows that $|G : X| > \mathrm{d}(G)$, a contradiction. Therefore $\bar{W}_x = \bar{V}$. Let $y \in \Gamma(X) - H_x$. Then H_1 normalizes $C_{\bar{V}}(y)$ so H_1 is not irreducible on $\bar{V}$. Also, $C_{\bar{V}}(H_1) \leqq \bar{Y}_x = 0$. Suppose $\bar{U}$ is a proper H_1-invariant subspace of $\bar{V}$ and $\bar{U}$ is non-degenerate, so $\bar{V} = \bar{U} \oplus \bar{U}^\perp$. $[\bar{U}, x] \neq 0$, so $\bar{U} \cap \bar{V}_x \neq 0$. If $\bar{V}_x \nleqq \bar{U}$, then $\bar{U} = \bar{V}$, contradicting $\bar{U} \neq \bar{V}$. So $\bar{U} \cap \bar{V}_x$ is a one-space. Now, let $y \in H_x$, $x \sim y$. Then $[\bar{U} \cap \bar{V}_x, y] = \langle \bar{U} \cap \bar{V}_x, \bar{U} \cap \bar{V}_y \rangle < (\bar{U} \cap \bar{V}_y)^\perp$. But if $\bar{W} = \bar{U}^\perp \cap \bar{V}_y = [\bar{U}^\perp, y]$, then $\bar{U} \cap \bar{V}_x \nleqq \bar{W}^\perp$, for if $\bar{U} \cap \bar{V}_x \leqq \bar{W}^\perp$, then $\bar{U} \cap \bar{V}_x \leqq \bar{V}_x \cap \bar{V}_y^\perp = 0$, a contradiction. Therefore the only non-degenerate H_1-invariant subspaces of $\bar{V}$ is $\bar{V}$. Now let $\bar{U}$ be a minimal H_1-invariant subspace of $\bar{V}$. $\operatorname{Rad} \bar{U}$ is H_1-invariant. If $\operatorname{Rad} \bar{U} = 0$, then $\bar{U}$ is non-degenerate, contrary to what we have just shown.

Thus $\operatorname{Rad}\bar{U} = \bar{U}$ and $\bar{U}$ is totally isotropic. Now if $y \in \Gamma(X) - H_x$, then $\bar{U} \cap C_{\bar{V}}(y)$ is H_1-invariant, so $\bar{U} \cap C_{\bar{V}}(y) = \bar{U}$ or 0. Suppose for some $y \in \Gamma(X) - H_x$, $\bar{U} \cap C_{\bar{V}}(y) = 0$. Since $\dim \bar{V}/C_{\bar{V}}(y) = 2$, $\dim \bar{U} = 2$. However, we then have $H_1 \leqq \mathrm{SL}(\bar{U}) \cong \mathrm{SL}_2(q)$ and then $\dim \bar{W}_x = \dim \bar{V} = 4$, a contradiction. So for all $y \in \Gamma(X) - H_x$, $[\bar{U}, y] = 0$. Then $[\bar{U}, H(y)] = 0$. But now with $H(y)$ in the role of H_1, we must have $C_{\bar{V}}(H(y)) = 0$, and this final contradiction completes (4.6).

Now suppose $G_W = Q \cdot L$ is a parabolic subgroup of G, where $Q = O_p(G_W)$, $Q \cap L = 1$. Set $L_1 = O^{p'}(L)$ and $P = QL_1$. Since P contains a p-Sylow of G, we may assume $U \cap X \in \mathrm{Syl}_p(X)$ where $U \in \mathrm{Syl}_p(P)$. Now clearly $|PX| \leqq |G|$. Since $|PX| = |P| \cdot |X|/|P \cap X|$ we have

(4.7) $$\mathrm{d}(G) > |G : X| \geqq |P : P \cap X|.$$

Now $|P : P \cap X| = |P : Q(P \cap X)| \cdot |Q(P \cap X) : P \cap X|$. $|P : Q(P \cap X)| = |P/Q : Q(P \cap X)/Q|$, $|Q(P \cap X) : P \cap X| = |Q : Q \cap X|$. Therefore

(4.8) $$\mathrm{d}(G) > |P/Q : Q(P \cap X)/Q| \cdot |Q : Q \cap X|.$$

Note that $P/Q \cong L_1$. This will be a linear or classical group which we have already established the result for, and so we will have a lower bound for $|P/Q : Q(P \cap X)/Q|$. We will use this in subsequent sections to show $\Gamma(X) \neq \varnothing$.

§ 5. Linear groups

The result for $G = \mathrm{SL}_n(q)$ is really quite easy.

(5.1) LEMMA. $n = \dim V \geqq 4$.

PROOF. All subgroups of $\mathrm{SL}_n(q)$ are known for $n \leqq 3$ (see [4], [7], [8], [13]) and the result holds in these cases.

(5.2) LEMMA. *X is not flag-transitive.*

PROOF. From the list of groups with a proper flag-transitive subgroup in [16] we see $|G : X| \geqq \mathrm{d}(G)$, contrary to assumption.

(5.3) LEMMA. *X is transitive on the points and lines of* $\mathrm{PG}(n-1, q)$.

PROOF. Let π be the permutation character of G on $\mathrm{PG}_1(n-1, q)$ and τ be permutation character on $\mathrm{PG}_2(n-1, q)$. $\pi - 1_G$ and $\pi - \tau$ are irreducible characters for G. The degree of $\pi - \tau >$ degree $\pi - 1_G = q(q^{n-1} - 1)/(q-1)$. Therefore $(\pi - 1_F, 1_X^G) = (\pi - \tau, 1_X^G) = 0$ since $|G : X| < \mathrm{d}(G) \leqq (q^n - 1)/(q-1)$. Thus $(\pi, 1_X^G) = (\tau, 1_X^G)$ from which the result follows.

(5.4) LEMMA. *X contains a root subgroup of G.*

PROOF. Let W be a one-space of V, $G_W = Q \cdot L, L_1, P$ as in §4. Clearly we may assume $Q \not\subseteq X$. Suppose $P = Q(P \cap X)$. Then $P \cap X$ acts irreducibly on Q. However, $P \cap X$ normalizes $Q \cap X$, so since $Q \cap X \neq Q$, $Q \cap X = 1$. Then since $U \cap X \in \mathrm{Syl}_p(X)$, q^{n-1} divides $|G : X|$. However, $2q^{n-1} > (q^n - 1)/(q-1) \geqq \mathrm{d}(G)$. Therefore $|G : X| = q^{n-1}$. But then $UX = G$. Thus if $B = N_G(U)$, also $BX = G$, and X is flag-transitive, contrary to (5.2). So $P \neq Q(P \cap X)$. Moreover, by the argument above $Q \cap X \neq 1$ and $P \cap X \cong Q(P \cap X) \cap L$ does not act irreducibly on Q. Note we are considering Q as an $\mathbf{F}_q$-space for $L_1 \cong \mathrm{SL}_{n-1}(q)$. Suppose that L_2 normalizes a k-space of Q, with $2 \leqq k \leqq n-3$. So

$$\frac{q^n - 1}{q - 1} \geqq \frac{(q^{n-1} - 1)(q^{n-2} - 1)}{(q^2 - 1)(q - 1)}.$$

But since $2 \leqq n \leqq 3$, $n \geqq 5$, and this leads to a contradiction. So L_1 normalizes a unique subspace of Q and is either a point of Q, or a hyperplane. In either case $Q \cap X$ contains a root subgroup as claimed.

(5.5) THEOREM. *X does not exist and the result holds for $G = \mathrm{SL}_n(q)$.*

PROOF. By [11, 12] since X contains root subgroups and is transitive on the points of $\mathrm{PG}(n-1, q)$, $X \cong \mathrm{Sp}(V)$ or $\mathrm{SL}(V)$. As $X < G \cong \mathrm{SL}(V)$, $X \cong \mathrm{Sp}(V)$. But X is transitive on lines of $\mathrm{PG}(n-1, q)$ by (5.13), however, $\mathrm{Sp}(V)$ is not transitive on lines of $\mathrm{PG}(n-1, q)$ and the theorem is proved.

§6. Symplectic groups for fields of odd characteristic

(6.1) LEMMA. $n = 2m \geqq 6$.

PROOF. By [14] the minimal degree for $G = \mathrm{Sp}_4(q)$, q odd is $\mathrm{d}(G)$.

(6.2) LEMMA. $\Gamma(X) \neq \varnothing$.

PROOF. We assume on the contrary that $\Gamma(X) = \varnothing$ and derive a contradiction. Let W be a one-space of V, $G_W = QL, U, L_1, P$ as in §4. Also set $B = N_G(U)$. Since q is odd, Q is a special group. $|Q| = q^{2n-1}$, $Q_0 = Z(Q) = \Phi(Q) = Q' \in \mathscr{X}$, $L_1 \cong \mathrm{Sp}_{2m-2}(q)$ and Q/Q_0 is a Symplectic module for L_1. Since $\Gamma(X) = \varnothing$, $Q_0 \cap X = 1$. Consequently $Q \cap X$ is elementary abelian. The maximal elementary abelian subgroups of Q all contain Q_0 and map onto maximal isotropic subspaces of Q/Q_0. Therefore $|Q \cap X| \leqq q^{m-1}$ and

(6.3) $$|Q : Q \cap X| \geqq q^m.$$

Suppose $Q \cap X = 1$. Then q^{2m-1} divides $|G : X|$. Since $2q^{2m-1} > q^{2m} - 1/q - 1 = \mathrm{d}(G)$, $q^{2m-1} = |G : X|$. Then $BX = UX = G$ and X is flag-transitive. By [16], $X \geqq \langle U^G \rangle = G$, a contradiction. So $Q \cap X \neq 1$. If $Q(P \cap X) = P$, then $P \cap X$ is irreducible on Q/Q_0. However, $Q_0(Q \cap X)/Q_0 \neq 1$, Q/Q_0, so $P \cap X$ cannot act irreducibly on Q/Q_0. Therefore $Q(P \cap X) < P$, so $|P/Q : Q(P \cap X)/Q| \neq 1$. Set $L_2 = Q(P \cap X) \cap L$, so $|P/Q : Q(P \cap X)/Q| = |L_1 : L_2|$. By induction we have a lower bound for $|L_1 : L_2|$. Suppose $m \geqq 4$. Then

(6.4) $$|L_1 : L_2| \geqq \frac{q^{2m-2} - 1}{q - 1}.$$

Now by (6.3), (6.4), and (4.8) we have

(6.5) $$\frac{q^{2m} - 1}{q - 1} > \frac{q^m (q^{2m-2} - 1)}{q - 1}.$$

Thus $q^{2m} > q^m (q^{2m-2} - 1)$, so also $q^{2m} > q^m (q^{2m-2} - 1)$ or $q^m > q^{2m} - 1$. It then follows that $q^m \geqq q^{2m-2}$., so $m \geqq 2m - 2$ or $m \leqq 2$ contradicting our assumption that $m \geqq 4$. Therefore $m = 3$. Suppose $q > 3$. Then $|L_1 : L_2| \geqq q^3 + q^2 + q + 1$. On the other hand by (6.3) $|Q : Q \cap X| \geqq q^3$. Now by (4.8) we have

$$\frac{q^6 - 1}{q - 1} > q^3(q^3 + q^2 + q + 1),$$

a contradiction. Therefore $m = 3$, $q = 3$. But now $|L_1 : L_2| \geqq 27$ and $|Q : Q \cap X| \geqq 27$, so by (4.8) $(3^6 - 1)/2 > 3^3 \cdot 3^3$, a contradiction and the lemma is proved.

Now let $H = \langle \Gamma(X) \rangle$. By (4.5) $\Gamma(X)$ is a conjugate class in H.

(6.6) LEMMA. *H acts irreducibly on V.*

PROOF. Note that X normalizes $C_V(H)$, so $C_V(H) = 0$ since by (4.1) X acts irreducibly on V. Let W be an H-invariant subspace of V. Then since $[W, H] \neq 0$, $[W, X] \neq 0$ for all $x \in \Gamma(X)$. Then $V_x \leqq W$, so $W \geqq \langle V_x : x \in \Gamma(X) \rangle$. Clearly X normalizes $\langle V_x : x \in \Gamma(X) \rangle$, so $\langle V_x : x \in \Gamma(X) \rangle = V$ and $W = V$ as asserted.

(6.7) THEOREM. *The result holds for $G = \mathrm{Sp}_{2m}(q)$, q odd.*

PROOF. By [9], the only possibility for H is $\mathrm{Sp}_{2m}(q_0)$ where $q_0 = p^{e_0}$, $e_0 | e$. Then $H = N_G(H)$ and simple arithmetic yields $|G : X| > \mathrm{d}(G)$, a contradiction.

§7. Orthogonal groups in odd dimension

In this section we prove the result for orthogonal groups in odd dimension and include the case of characteristic two. Since $\Omega_{2m+1}(2^e) = \mathrm{Sp}_{2m}(2^e)$ the result will then be proved for these groups as well.

(7.1) LEMMA. $n = 2m + 1 \geqq 7$.

PROOF. $\Omega_5(q) \cong \mathrm{Sp}_4(q)$. For q odd the result is in [14]. For q even the result can be found in [5].

(7.2) LEMMA. $\Gamma(X) \neq \varnothing$.

PROOF. We assume on the contrary that $\Gamma(X) = \varnothing$ and get a contradiction. We first treat the case of odd q. Let W be a singular one-space of V, $G_W = Q \cdot L$, L_1, P, U as in §4, and also set $B = N_G(U)$. Q is elementary abelian of order q^{2m-1}, $L_1 \cong \Omega_{2m-1}(q)$ acts irreducibly on Q, and as a module for L_1, Q is the orthogonal module. The "singular" vectors in Q are long root elements. There are subgroups of Q of order q^{m-1} all of whose non-identity elements are long root elements, namely the maximal totally singular subgroups of Q. Consequently $|Q \cap X| \leqq q^m$ and $|Q : Q \cap X| \geqq q^{m-1}$. If $Q \cap X = 1$, then q^{2m-1} divides $|G : X|$. But $2q^{2m-1} > (q^{2m-1}/(q-1)) = \mathrm{d}(G)$, so that we would then have $q^{2m-1} = |G : X|$. Then $BX = UX = G$ and X is flag-transitive. By [16] we then have $X \geqq \langle U^G \rangle = G$, a contradiction. So $Q \cap X \neq 1$. Since $Q \cap X < Q$ and $P \cap X$ normalizes $Q \cap X$, $Q(P \cap X) \neq P$. If $L_2 = Q(P \cap X) \cap L_1$, then $|L_1 : L_2| = |P \cap X : Q(P \cap X)| > 1$. We can apply induction to L_1 to get a lower bound for $|L_1 : L_2|$. Suppose $m \geqq 4$. Then $|L_1 : L_2| \geqq (q^{2m-2} - 1)/(q-1)$ and since $|Q : Q \cap X| \geqq q^{m-1}$, by (4.8) we get

$$\mathrm{d}(G) > \frac{q^{m-1}(q^{2m-2}-1)}{q-1}. \tag{7.3}$$

Since $\mathrm{d}(G) = (q^{2m} - 1)/(q-1)$ this leads to the inequality $q^{2m} - 1 > q^{m-1}(q^{2m-2} - 1)$ or $q^{2m} > q^{m-1}(q^{2m} - 1)$. Then $q^{m+1} > q^{2m-2} - 1$ or $q^{m+1} \geqq q^{2m-2}$, so $m + 1 \geqq 2m - 2$ and $3 \geqq m$ contrary to our assumption that $m \geqq 4$. Thus $m = 3$. Suppose $q > 3$. Then $|L_1 : L_2| \geqq (q^4 - 1)/(q-1)$. On the other hand $|Q : Q \cap X| \geqq q^2$, so $(q^6 - 1)/(q-1) > q^2 |L_1 : L_2|$ by (4.8), and this gives

$$q^3 + q^2 + q + 2 > |L_1 : L_2| \qquad \text{or} \qquad q^3 + q^2 + a + 1 \geqq |L_1 : L_2|.$$

Therefore we have equality. By [14] there are two possibilities for the class of L_2 in L_1: the stabilizer of a "singular point" of Q or the stabilizer of a "singular line" of Q. In either case, if $R = O_p(L_2)$, then $C_Q(R)$ is a singular subgroup of Q. However, since $|L_1 : L_2| = (q^4 - 1)/(q - 1)$ we must have $|Q \cap X| \geqq q^3$ from (4.8), and therefore $|Q \cap X| = q^3$. L_2 and R normalizes $Q \cap X$. Then $C_{Q\cap X}(R) \neq 1$. But $\mathscr{E}_1(C_{Q\cap X}(R)) \subseteq \mathscr{E}_1(C_Q(R)) \subseteq \Gamma$, contradicting $\Gamma(X) = \varnothing$. Therefore $q = 3$. Now $|L_1 : L_2| \geqq 27$ and from (4.8) we get $364 > |Q : Q \cap X| \cdot 27$. Then $|Q : Q \cap X| < 14$, so $|Q : Q \cap X| \leqq 9$, so $|Q \cap X| = 27$. However, Q is an orthogonal space over $\mathbf{F}_3$ and any 3-subspace must contain singular vectors. Then $\varnothing \neq \Gamma(X \cap Q) \subseteq \Gamma(X) = \varnothing$, and this contradiction completes the case q odd.

Now assume q even. Let W be a singular one-subspace of V, $G_W = Q \cdot L$, L_1, P, U, B as before. $L_1 \cong \Omega_{2m-1}(q)$ and Q is again elementary of order q^{2m-1}. However, $Q_0 = Z(P)$ has order q, is a full transvection group on $V/\mathrm{Rad}\, V$, and L_1 acts irreducibly as $\mathrm{Sp}_{2m-2}(q)$ on Q/Q_0. Q again contains subgroups Q_1 of order q^{m-1} so $\mathscr{E}_1(Q_1) \subseteq \Gamma$, and so $|Q \cap X| \leqq q^m$ and $|Q : Q \cap X| \geqq q^{m-1}$. Set $L_2 = Q(P \cap X) \cap L_1$, so $|P : Q(P \cap X)| = |L_1 : L_2|$. If $L_2 \geqq L_1'$ (i.e., if $Q(P \cap X)$ cover L_1'), then we must have $Q \cap X \leqq Q_0$ and q^{2m-2} divides $|Q : Q \cap X|$ and $|G : X|$. Note when $q = 2$, $2^{2m-1} > 2^{m-1}(2^m - 1) = d(G)$, and so in this case $|G : X| = 2^{2m-2}$. But then X is flag-transitive, and then by [16], $X = G$, a contradiction. Therefore $q > 2$. Note now that we must have $|G : X| = q^{2m-2}a$ and $a \leqq q + 1$. Also $O^\infty(P \cap X/Q \cap X) \cong \Omega_{2m-1}(q)$. Since $q \geqq 4$, $\Omega_{2m-1}(q)$ does not have a double cover, so $O^\infty(P \cap X) \cong \Omega_{2m-1}(q)$. Now for any maximal totally singular subspace W_1 containing W, $N_{P\cap X}(W_1) \geqq \mathrm{SL}(W_1/W)$. Consider $G_{W_1} = R \cdot K$ where $R = O_2(G_{W_1})$, $R \cap K = 1$. Set $K_1 = O^{2'}(K)$, $P_1 = RK_1$. q^{2m-2} divides $|P_1 : P_1 \cap X|$ and $q^{2m-2}a = |G : X| \geqq |P_1 : P_1 \cap X| = |R : R \cap X| \cdot |K_1 : K_2|$ where $K_2 = R(P_1 \cap X) \cap K_1$, by (4.8). We remark that R is elementary of order $q^{\frac{1}{2}m(m+1)}$ and is indecomposable as a module for K_1. It has a unique proper K_1-invariant subgroup R_0 of order $q^{\frac{1}{2}m(m-1)}$. As a module for K_1, $R_0 \cong \Lambda^2(W)$ and $R/R_0 \cong W^* = \mathrm{Hom}_{\mathbf{F}_q}(W, \mathbf{F}_q)$. Now $|R| > d(G)$, so $R \cap X \neq 1$. Since K_1 acts indecomposably on $R \cap X$, if $K_2 = K_1$, then $R_0 \leqq X$ which would imply $\Gamma(X) \neq \varnothing$ since R_0 contains subgroups R_1 of order q^{m-1} such that $\mathscr{E}_1(R_1) \subseteq \Gamma$. Therefore $K_1 > K_2$. Since $m \geqq 3$, $q > 2$, $|K_1 : K_2| \geqq q^m - 1/(q - 1)$. On the other hand $|R : R \cap X| \geqq q^{m-1}$. From this it follows that $|K_1 : K_2| = 2^b r$ where $(2, r) = 1$, $r \leqq q + 1$ and $q^{m-1} \leqq 2^b < q^m$. Therefore K_2 must act irreducibly on W_1. Now let $K_3 = N_{K_1}(W)$, $P_2 = N_{P_1}(W)$. We saw that $P_2 \cap X = N_X(W_1)$ induces $\mathrm{SL}_{m-1}(q)$ on W_1/W. Consequently $K_2 \cap K_3$ covers $O^{2'}(K_3/O_2(K_3))$. If $K_2 \cap O_2(K_3) \neq 1$, then $O_2(K_3) \leqq K_2$ and then $(|K_1 : K_2|, 2) = 1$. Therefore $K_2 \cap O_2(K_3) = 1$. But then $(K_2 \cap K_3)' \cong \mathrm{SL}_{m-1}(q)$. Since $q > 2$, if $m > 3$, then by [10] $(K_2 \cap K_3)'$ splits over W, and hence contains full root subgroups. But then by [7], $K_2 = K_1$, a contradiction. Therefore $m = 3$. However all subgroups of K_1 are known and there is no such K_2. So we have a contradiction. Therefore we cannot have $L_2 \geqq L_1'$, and we have also shown that $Q \cap X \not\leqq Q_0$. Therefore since L_2 normalizes $Q \cap X \neq Q_0, Q$ we must in fact have $|L_1 : L_2| \geqq (q^{2m-2} - 1)/(q - 1)$, even when $q = 2$. Since $|Q : Q \cap X| \geqq q^{m-1}$ we have either

$$\frac{q^{2m}-1}{q-1} > q^{m-1}\frac{(q^{2m-2}-1)}{q-1} \qquad \text{when } q > 2,$$

or

$$2^{m-1}(2^m - 1) > 2^{m-1}(2^{2m-2} - 1) \qquad \text{when } q = 2.$$

The second case leads to $2 > m$ which is not the case, so $q > 2$. Thus $q^{2m} > q^{m-1}(q^{2m-2} - 1)$ or $q^{m+1} > q^{2m-2} - 1$, and so $m + 1 \geqq 2m - 2$ and $m \leqq 3$. Therefore $m = 3$. Now $(q^6 - 1)/(q - 1) > q^2|L_1 : L_2|$ so $q^3 + q^2 + q + 2 > |L_1 : L_2|$, so we must have $|L_1 : L_2| = (q^4 - 1)/(q - 1)$. By [5], there are two possibilities for the class of L_2. In either case L_2 normalizes a unique minimal subgroup of Q containing Q_0, and in either case it contains long root elements. Since L_2 normalizes $Q \cap X \neq Q_0$, $\Gamma(Q \cap X) \neq \varnothing$, and this contradiction completes the lemma.

Now let $H = \langle \Gamma(X) \rangle$.

(7.3) LEMMA. *Let $\bar{V} = V/\text{Rad}\, V$ and denote images in $\bar{V}$ by $\bar{\ }$. H does not normalize any proper non-degenerate subspace of $\bar{V}$.*

PROOF. This is contained in the proof of (4.6).

(7.4) LEMMA. *Either H is irreducible on $\bar{V}$ or q is even and H normalizes a pair of maximal totally isotropic subspaces of $\bar{V}$ which intersect in 0.*

PROOF. Let $\bar{W}$ be a minimal H-invariant subspace of $\bar{V}$. If $\bar{W} \neq \bar{V}$, then $\bar{W}$ is totally singular if $V = \bar{V}$ or totally isotropic if Rad $V \neq 0$, since Rad $\bar{W} \neq 0$ if $\bar{W} \neq \bar{V}$ by (7.3). Let $g \in X$ so $\bar{W} \neq \bar{W}^g$. Such a g exists since X is irreducible on $\bar{V}$ by (4.1). Then $\bar{W} + \bar{W}^g = \bar{W} \oplus \bar{W}^g$ must be non-degenerate, and so by (7.3) $\bar{V} = \bar{W} \oplus \bar{W}^g$, and dim $\bar{V}$ is even, so that $\bar{V} \neq V$ and q is even. Note in the latter case that if $\bar{U}$ is a maximal totally isotropic subspace of $\bar{V}$ normalized by H, then for any $x \in \Gamma(X)$ $V_x \cap \bar{U} \neq 0$ and if $V_x \cap \bar{U} = V_x \cap \bar{W}$, then $\bar{U} = \bar{W}$. This implies that H normalizes at most $q + 1$ maximal totally isotropic subspaces of $\bar{V}$.

(7.5) LEMMA. *H is irreducible on $\bar{V}$.*

PROOF. From (7.4) we can assume q is even, H not irreducible on $\bar{V}$. Let $\bar{W}_1, \cdots, \bar{W}_t$ be the maximal totally isotropic subspaces normalized by H, $t \leqq q+1$. X permutes these, so $|X : N_X(\bar{W}_1)| \leqq t$. Now if $R = O_p(N_X(\bar{W}_1))$, then $[R, H] \leqq R \cap H \leqq O_p(H) \leqq O_p(X) = 1$, so $R \leqq C_G(H)$. But H acts irreducibly on $\bar{W}_1$, R normalizes $\bar{W}_1$, so $C_{\bar{W}_1}(R) \neq 0$ and therefore $[R, \bar{W}_1] = 0$. Therefore $R \leqq C_G(\bar{W}_1) = Q$. Now Q is as described earlier, an elementary group of order $q^{\frac{1}{2}m(m+1)}$ with a subgroup of order $q^{\frac{1}{2}m(m-1)}$, so that if $\tilde{L} = O^{2'}(N(\bar{W}_1)/C(\bar{W}_1)) \cong SL(\bar{W}_1)$, then $Q_0 \cong \Lambda^2(\bar{W}_1)$ and $Q/Q_0 \cong \bar{W}_1^* = \mathrm{Hom}_{\mathbf{F}_q}(\bar{W}_1, \mathbf{F}_q)$ as modules for $\tilde{L}$. Since H acts irreducibly on $\bar{W}_1$, $R \leqq Q_0$. Therefore

$$|G : N_X(\bar{W}_1)| \geqq q^m \cdot \prod_{j=1}^{m} (q^j + 1).$$

Now it follows that

$$|G : X| \geqq q^m \prod_{j=1}^{m} (q^j + 1)/t \geqq q^m \cdot \prod_{j=2}^{m} (q^j + 1) \geqq q^m (q^m + 1) > \mathrm{d}(G),$$

a contradiction.

(7.6) THEOREM. *The result holds for* $G = \Omega_{2m+1}(q)$.

PROOF. By the main theorem in [9] we can identify H. $H \cong \Omega_{2m+1}(q_0)$, $q_0 = p^{e_0}$, $e_0 | e$ or $H \cong G_2(q_0)$, $q_0 = p^{e_0}$, $e_0 | e$ and $m = 3$. In either case clearly $|G : N_G(H)| > \mathrm{d}(G)$, which yield a contradiction.

§8. $\Omega^+_{2m}(q)$

Because of the isomorphisms $\Omega^+_4(q) \cong SL_2(q) \times SL_2(q)$, $\Omega^+_6(q) \cong SL_4(q)$, we need only consider the cases $m \geqq 4$.

(8.1) LEMMA. $\Gamma(X) \neq \varnothing$.

PROOF. Assume $\Gamma(X) = \varnothing$. Let W be a singular one-subspace of V, $G_W = QL, L_1, P, U, B$ as in previous sections. Q is an elementary abelian group of order q^{2m-2} and an orthogonal modules for $L_1 \cong \Omega^+_{2m-2}(q)$. If $x \in Q$ is a singular vector, then $\langle x \rangle \in \Gamma$. Since the maximal singular subspaces of Q have order q^{m-1}, $|Q : Q \cap X| \geqq q^{m-1}$. Set $L_2 = (P \cap X) \cap L_1$. If $L_2 = L_1$, then we must have $Q \cap X = 1$. Then since $U \cap X \in \mathrm{Syl}_p(X)$, q^{2m-2} divides $|G : X| < \mathrm{d}(G) < 2q^{2m-2}$. Then $q^{2m-2} = |G : X|$. This implies $BX = UX = G$, X is flag-transitive. By [16], $X \geqq \langle U^G \rangle$, a contradiction. Therefore $Q \cap X \neq 1$. Consequently $L_2 \neq L_1$. From (4.8) $\mathrm{d}(G) > |Q : Q \cap X| \cdot |L_1 : L_2|$. Suppose $m \geqq 5$, $q > 2$. Then

$$\frac{(q^m-1)q^{m-1}+1}{q-1} > q^{m-1}\frac{(q^{m-1}-1)(q^{m-2}+1)}{q-1}$$

by induction. Then $q^{2m-1}+q^m-q^{m-1}+1 > q^{m-1}(q^{2m-3}+q^{m-1}-q^{m-2}-1)$, so also $q^{2m-1}+q^m+q^{m-1} > q^{m-1}(q^{2m-3}+q^{m-1}-q^{m-2}-1)$, thus $q^m+q-1 > q^{2m-3}+q^{m-1}-q^{m-2}-1$ or

$$q^{m-1}+1 > q^{2m-3}+q^{m-1}-q^{m-2} > q^{2m-3}+q^{m-2} > q^{2m-3}+1.$$

Therefore $m-1>2m-3$ so $m<4$ contradicting $m \geqq 5$. Suppose $q=2$, $m \geqq 5$. Then $2^{m-1}(2^m-1) > 2^{m-1}\cdot[2^{m-2}(2^{m-1}-1)]$ which yield $2^m-1 > 2^{m-2}(2^{m-1}-1)$, from which it follows that $4 > 2^{m-1}-1 \geqq 15$, a contradiction. Therefore $m=4$. Suppose $q>2$. Then from (4.8) we have

$$\frac{(q^3+1)(q^4-1)}{q-1} > q^3|L_1:L_2|,$$

so $q^3+q^2+q+2 \geqq |L_1:L_2|$. However, q^3+q^2+q+2 does not divide $|\mathrm{SL}_4(q)|$, so in fact $q^3+q^2+q+1 \geqq |L_1:L_2| \geqq (q^4-1)/(q-1)$, so we have equality. From this it follows that $|Q \cap X| = q^3$. There are two possibilities for the class of L_2 in L_1. In either case L_2 normalizes a unique proper subgroup of Q, $C_Q(O_p(L_2))$ and this is a totally singular subgroup of Q. However, L_2 normalizes $Q \cap X$, so $\mathscr{E}_1(Q \cap X) \subseteq \Gamma$, contradicting $\Gamma(X) = \varnothing$. So it remains to consider $m=4$, $q=2$. Now from (4.8) we have $2^3(2^4-1) > 2^3|L_1:L_2|$, so $|L_1:L_2| < 15$. Since $L_1 \cong \mathrm{SL}_4(2) \cong A_8$ we must have $L_2 \cong A_7$. However, then L_2 is irreducible on Q and this contradicts $1 \neq Q \cap X < Q$.

(8.2) THEOREM. *The results hold for* $\Omega^+_{2m}(q)$.

PROOF. We first prove that $H = \langle \Gamma(X) \rangle$ acts irreducibly on V. By the proof of (4.6) the only non-degenerate subspace of V normalized by H is V. Suppose W is a minimal H-invariant subspace of V. Since $[V,H] \neq 0$, X is irreducible and normalizes $C_V(H)$, so $C_V(H)=0$. Rad $W=0$, so W is totally singular. Since $H = \langle \Gamma(X) \rangle$, $W = \langle [W,X] : X \in \Gamma(X) \rangle$. Let W^g be an X-conjugate of W. If $x \sim y$ in $\Gamma(X)$, then $V_x \cap W \not\leqq (V_y \cap W^g)^\perp$, so $W + W^g = W \oplus W^g$ is non-degenerate and hence $V = W + W^g$. As in (7.5) H normalizes at most $q+1$ maximal totally singular subspaces and so there are at most $q+1$ X-conjugates of W. Therefore $|X : N_X(W)| \leqq q+1$. Let $R = O_p(N_X(W))$. Then $[R,H]=1$, so we must have $[W,R]=0$. Let $Q = C_G(W) = O_p(N_G(W))$. As a module for $N_G(W)/C_G(W)$, $Q \cong \Lambda^2(W)$. Thus if $x \sim y \in \Gamma(X)$, then $|C_Q(\langle x,y \rangle)| = q^{\frac{1}{2}(m-2)(m-3)}$, and so we must have $|Q : C_G(H)| \geqq |Q : C_Q(\langle x,y \rangle)| = q^{2m-3}$. Therefore $|G : N_X(W)| \geqq q^{2m-3} \cdot |G : N_G(W)| > q^{2m-3}\mathrm{d}(G)$. But then $|G : X| >$

$q^{2m-3}\mathrm{d}(G)/(q+1) > \mathrm{d}(G)$, a contradiction. So our assertion is proved. Now by the main result in [9] we can identify H:

(i) $H = \Omega_{2n}^{+}(q_0)$, $q_0 = p^{e_0}, e_0 \mid e$. These are the centralizers of field automorphisms in G. Then $X = H$. The worst case is when $q = q_0^2$. Then $|G:X| = |G:H| > q^{\frac{1}{2}m(m-1)+16} > \mathrm{d}(G)$.

(ii) $H = \Omega_{2m}^{-}(q_0)$, $q_0 = p^{e_0}, e_0 \mid e/2$. As in (i) the most extreme case is when $q = q_0^2$. Then $|G:H| \geqq q^{\frac{1}{2}m(m-1)+10} > \mathrm{d}(G)$.

(iii) m is even, $H = \mathrm{SU}_n(q_0)$, $q_0 = p^{e_0}, e_0 \mid e$. Then $|G:N_G(H)| \geqq q^{\frac{1}{2}m(3m-4)} > \mathrm{d}(G)$.

(iv) $m = 4$, $H/Z(H) = P\Omega_7(q_0)$, $|Z(H)| = (2, q_0 - 1)$ where $q_0 = p^{e_0}, e_0 \mid e$. This embedding is conjugate in $\mathrm{Aut}\, P\Omega_8^{+}(q)$ to a group stabilizing a non-singular one-space. There are two such embeddings. The worst case is when $q_0 = q$. Then $|G:N_G(H)| = q^3(q^4-1)/2 > \mathrm{d}(G)$ when $q > 2$, and equals $\mathrm{d}(G)$ when $q = 2$.

(v) $m = 4$, $H = {}^3D_4(q_0)$, $q_0 = p^{e_0}, e_0 \mid e/3$. Worse case is $q^3 = q$. Then $|G:N_G(H)| > q^{24} > \mathrm{d}(G)$.

(vi) q is even, $m = 6$, $H \cong 3 \cdot P\Omega_6^{-,\pi}(3) < \mathrm{GU}_6(2) < \Omega_{12}^{+}(2)$; it is clear that this index is too great.

(vii) q even, $H = A \propto S_{2k}$ in $\Omega^{+}(4k, q)$, A is homocyclic group and $\exp(A) \mid q+1$. In all cases $H = N(H)$ and clearly $|G:N_G(H)| > \mathrm{d}(G)$.

(viii) q is even, $H = A \propto W$, A homocyclic with $\exp(A) \mid q-1$ and W the Weyl group of G. Then $H = N(H)$, and $|G:H| > \mathrm{d}(G)$.

§9. Unitary groups

Since $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$, and all subgroups of $\mathrm{SU}_3(q)$ are known [13], we may assume $n \geqq 4$. As in previous sections, we prove

(9.1) Lemma. $\Gamma(X) \neq \varnothing$.

Proof. Assume on the contrary that $\Gamma(X) = \varnothing$. Let W be an absolute point of V, $G_W = Q \cdot L$, L_1, P, U, B, L_2 as in previous sections. $Q_0 = Z(Q) = Q' = \Phi(Q) \geqq \mho'(Q)$, Q is special, $|Q| = q^{2n-3}$, and Q/Q_0 is the unitary module for $L_1 \cong \mathrm{SU}_{n-2}(q)$. Note that $\mathscr{E}_1(Q_0) \subseteq \Gamma$ and therefore $Q \cap X$ is elementary abelian. The maximal elementary abelian subgroups of Q correspond to the maximal totally isotropic subspaces of Q/Q_0. Therefore, if n is even, they have order q^{n-1}, and if n is odd they have order q^{n-2}. Since $Q_0 \cdot (Q \cap X)$ is elementary and $Q_0 \cap X = 1$, $|Q_0 \cdot (Q \cap X)| = q\,|Q \cap X|$ and so we have $|Q \cap X| \leqq q^{n-2}$ if n is even, and $|Q \cap X| \leqq q^{n-3}$ if n is odd. By (4.8) $\mathrm{d}(G) < |Q : Q \cap X| \cdot |L_1 : L_2|$. Suppose n is even, $n \geqq 6$. Then

$$\frac{(q^n-1)(q^{n-1}+1)}{q^2-1} > q^{n-1}|L_1 : L_2|. \tag{9.2}$$

If $Q \cap X = 1$, then as in previous sections X is flag-transitive and we have a contradiction. From this it follows that $L_1 \neq L_2$. If $n > 6$, then from (9.2) and induction we have

$$\frac{(q^n-1)(q^{n-1}+1)}{q^2-1} > q^{n-1}\frac{(q^{n-2}-1)(q^{n-3}+1)}{q^2-1},$$

which gives an easy contradiction. Suppose $n = 6$. Then we have $|L_1 : L_2| \geqq (q^3+1)(q+1)$, and from (9.2) we get

$$\frac{(q^6-1)(q^5+1)}{q^2-1} > q^5(q^3+1)(q+1),$$

again impossible. Therefore if n is even, then $n = 4$. Now since $|Q : Q \cap X| \geqq q^3$, q^3 divides $|G : X|$. Set $|G : X| = q^3\mu$. Note that $q^3(q+2) > (q^3+1)(q+1) = d(G)$, so $\mu \leqq q+1$. Now $L_1 \cong SL_2(q)$. All subgroups of $SU_4(2)$ are known (see [3]), so we may assume $q > 2$. We have by (4.8) $q^3\mu \geqq q^3|L_1 : L_2|$, so $\mu \geqq |L_1 : L_2|$. If $q = 3, 5, 7$, or 11, then $q \leqq |L_1 : L_2| \leqq \mu \leqq q+1$ so $\mu = q$ or $q+1$. If $\mu = q$, then X is flag-transitive, and then $X = G$ which is not the case. If $q = 9$, then $6 \leqq \mu \leqq 10$. However, if $\mu < 10$, then $L_2 \cong A_5$ and then L_2 acts irreducibly on Q/Q_0. Since $|Q : Q \cap X| \geqq q^3 > q$, this would imply $Q \cap X \leqq Q_0$, and then $Q \cap X = 1$, which we have already seen is not true. Therefore in all cases $\mu = q+1$. Now let W_1 be a maximal isotropic subspace of V so that $B \leqq G_{W_1}$, and consider $G_{W_1} = R \cdot K$ where $R = O_p(Q_{W_1})$, $R \cap K = 1$. Set $R_1 = O^{p'}(R)$, $P_1 = RK_1$, $K_2 = R(P_1 \cap X) \cap R_1$. R_1 is an elementary abelian group of order q^4 and the orthogonal module for $K_1 \cong \Omega_4^-(q) \cong SL_2(q^2)$. Then we have q^3 divides $|P_1 : P_1 \cap X| = |R : R \cap X| \cdot |K_1 : K_2| \leqq q^3(q+1)$. Since $|Q_1| = q^4$, $Q_1 \cap X \neq 1$, and without loss of generality $Q_0 \subseteq Q_1$. Since K_1 acts irreducibly on Q_1 we may assume $K_1 \neq K_2$. All subgroups of K_1 are known, and if $K_1 \neq K_2$, then $|K_1 : K_2|_{p'} \geqq (q^2-1)/2$ whereas $|P_1 : P_1 \cap X|_{p'} \leqq q+1$. Therefore $(q^2-1)/2 \leqq q+1$, so $(q-1)/2 \leqq 1$ and $q \leqq 3$. Since $q \neq 2$, $q = 3$. Now $P_1 \cap X$ contains a five-Sylow of P_1, and a five-Sylow acts irreducibly on Q_1 and we have a contradiction.

Therefore, n is odd. Now from (4.8) we have

$$\frac{(q^n+1)(q^{n-1}-1)}{q^2-1} > q^n\,|L_1 : L_2|.$$

Once again we cannot have $Q \cap X = 1$, so $L_1 \neq L_2$, and hence by induction $|L_1 : L_2| \geqq (q^{n-2}+1)(q^{n-3}-1)/(q^2-1)$ and so

$$\frac{(q^n+1)(q^{n-1}-1)}{q^2-1} > q^n \frac{(q^{n-2}+1)(q^{n-3}-1)}{q^2-1}.$$

This gives $q^{2n-1}+q^n+q^{n+1}-1 > q^n(q^{2n-5}+q^{n-2}+q^{n-3}-1)$ and so $q^{2n-1}-q^n+q^{n-1} > q^n(q^{2n-5}-q^{n-2}+q^{n-3}-1)$. Then $q^n-q+1 > q(q^{2n-5}-q^{n-2}+q^{n-3}-1)$, and so $q^n-q \geqq q(q^{2n-5}-q^{n-2}+q^{n-3}-1)$ or $q^{n-1} > q^{2n-5}-q^{n-2}+q^{n-3}-1$. Finally

$$q^{n-1} \geqq q^{2n-5}-q^{n-2}+q^{n+3} \qquad \text{or} \qquad q^2 \geqq q^{n-2}-q+1 > q^{n-2}-q.$$

Thus $q > q^{n-3}-1$ which implies $n-3 \leqq 1$ and $n \leqq 4$ contrary to $n \geqq 5$, and with this contradiction the lemma is complete.

(9.3) Lemma. *$H = \langle \Gamma(X) \rangle$ is irreducible on V.*

Proof. Note that X normalizes $C_V(H)$, so $C_V(H)=0$ by (4.1). Let W be an H-invariant subspace of V. Then as $H = \langle \Gamma(X) \rangle$, $[W,H] \neq 0$, $[W,X] \neq 0$ for all $x \in \Gamma(X)$. Then $\langle V_x : x \in \Gamma(X) \rangle \leqq W$. But X normalizes $\langle V_x : x \in \Gamma(X) \rangle$, so $V = \langle V_x : x \in \Gamma(X) \rangle \leqq W$ and $V = W$ as asserted.

(9.4) Theorem. *The result is true for Unitary groups.*

Proof. By theorem II in [9], the possibilities for H are as follows:

(i) n is even, $H = \mathrm{Sp}_n(q_0)$, $q_0 = p^{e_0}, e_0 \mid e$,

(ii) $\mathrm{SU}_n(q_0)$, $q_0 = p^{e_0}, e_0 \mid e$,

(iii) Σ_{2n+1}, Σ_{2n+2} in $\mathrm{SU}_{2n}(q)$, q even,

(iv) $3 \cdot P\Omega_6^{-,\pi}(3)$ in $\mathrm{SU}_6(q)$, $q = 2^e$, e odd,

(v) $O_{2n}^{\pm}(q) < \mathrm{Sp}_{2n}(q) < \mathrm{SU}_{2n}(q)$, q even.

In all cases it is a routine calculation to show $|G : N_G(H)| > \mathrm{d}(G)$, and with this contradiction the theorem is established.

§10. $\Omega_{2n}^-(q)$

We are assuming $n \geqq 4$ since $\Omega_6^-(q) \cong \mathrm{SU}_4(q)$ and $\Omega_4^-(q) \cong \mathrm{SL}_2(q^2)$ and the results for these groups have already been established.

(10.1) Lemma. $\Gamma(X) \neq \varnothing$

Proof. Assume $\Gamma(X) = \varnothing$, and let W be a singular one-subspace of V. Let $G_W = QL$, L_1, P, U, B, L_2 be as in previous sections. Q is elementary abelian, $|Q| = q^{2m-2}$ and is the orthogonal module for $L_1 \cong \Omega_{2m-2}^-(q)$. The singular vectors of Q generate members of Γ. Since a maximal totally singular subspace of Q has order q^{n-2}, $|Q \cap X| \leqq q^n$. From (4.8) we have

$$\frac{(q^n+1)(q^{n-1}-1)}{q-1} > q^{n-2}|L_1 : L_2|.$$

As in previous sections, $Q \cap X = 1$ implies X flag-transitive and from [16] we get a contradiction. As in the previous sections this implies $L_1 \neq L_2$. Suppose $n > 4$. Then we have

$$\frac{(q^n+1)(q^{n-1}-1)}{q-1} > q^{n-2}\frac{(q^{n-1}+1)(q^{n-2}-1)}{q-1},$$

and this will lead to a contradiction. Therefore $n = 4$. Now let W be a singular two subspace of V and consider $G_{W_1} = RK$ where $R = O_p(G_{W_1})$, $R \cap K = 1$. Set $K_1 = O^{p'}(K)$, $P_1 = RK_1$, $K_2 = R(P_1 \cap X) \cap K_1$. $R_0 = R' = Z(R) = \Phi(R) \geqq \mho'(R)$ has order q. R is special of order q^9. $\mathscr{E}_0(Q_0) \subseteq \Gamma$. The maximal elementary subgroups of R of maximal order have order q^5 and of course contain R_0. Since $R_0 \cap X = 1$, $|R \cap X| \leqq q^4$ and so $|R : R \cap X| \geqq q^5$ and so q^5 divides $|G : X|$. Therefore

$$|G : X| = q^5\mu < \frac{(q^4+1)(q^3-1)}{q-1} = q^6 + q^5 + q^4 + q^2 + q + 1 < q^5(q+1).$$

So $\mu \leqq q + 1$. Now from the way in which (4.8) was derived we have

$$q^5(q+1) \geqq q^5\mu = |G : X| \geqq |Q : Q \cap X|\,|L_1 : L_2| \geqq q^2|L_1 : L_2|.$$

Therefore $|L_1 : L_2| \leqq q^3(q+1)$. But $L_1 \cong SU_4(q)$ and we showed in §9 that $|L_1 : L_2| \geqq (q^3+1)(q+1) > q^3(q+1)$ and this contradiction completes the proof of (10.1).

Set $H = \langle \Gamma(X) \rangle$. As in §§7 and 8 we must have

(10.2) Lemma. *H is irreducible on V.*

Now we can quote the main theorem in [9] as in the previous sections to get a contradiction. We omit the details because of the similarity to the previous sections. Thus we have

(10.3) Theorem. *The result is true for $\Omega_{2m}^-(q)$.*

References

1. M. Aschbacher and G. Seitz. *Involutions in Chevalley groups over fields of even order*, to appear.

2. B. Cooperstein, *Some geometries associated with parabolic representations of groups of Lie type*, Canad. J. Math. **27** (1976), 1021–1031.

3. L. E. Dickson, *Determination of all subgroups of the known simple group of order 25920*, Trans. Amer. Math. Soc. **5** (1904), 126–166.

4. L. E. Dickson, *Linear Groups With an Exposition of Galois Field Theory*, reprint of 1901 ed., Dover, New York, 1958.

5. D. E. Flesner, *Maximal subgroups of* $PSp_4(2^n)$ *containing central elations or noncentral skew elations*, to appear in Illnois J. Math.

6. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.

7. R. W. Hartley, *Determination of the ternary collineation groups whose coefficients lie in* $GF(2^n)$, Ann. of Math. **27** (1925), 140–158.

8. W. Huppert, *Endliche Gruppen*, Vol. 1, Springer Verlag, Berlin, 1967.

9. W. M. Kantor, *Subgroups of classical groups generated by long root elements*, unpublished.

10. J. E. McLaughlin, *Course Notes on Cohomology of Linear and Classical Groups*, unpublished.

11. J. E. McLaughlin, *Some groups generated by transvections*, Arch. Math. Basel **18** (1967), 364–368.

12. J. E. McLaughlin, *Some subgroups of* $SL_n(\mathbf{F}_2)$, Illinois J. Math. **13** (1969), 108–115.

13. M. H. Mitchell, *Determination of the ordinary and modular ternary lineary groups*, Trans. Amer. Math. Soc. **12** (1911), 207–242.

14. H. H. Mitchell, *The subgroups of the quaternary Abelian linear group*, Trans. Amer. Math. Soc. **15** (1914), 379–396.

15. W. Patton, Ph.D. Thesis, 1971.

16. G. M. Seitz, *Flag transitive subgroups of Chevalley groups*, Ann. of Math. **97** (1973), 27–56.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CALIF. 95064 USA